Quantum Features of Microwave Propagation in a Rectangular Waveguide

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Z. Naturforsch. **45a**, 953–957 (1990); received September 7, 1990

The formal analogy between the distribution of the electromagnetic field in waveguides and microwave cavities and quantum mechanical probability distributions is put into evidence. A waveguide of a cut-off frequency ω_c acts on an electromagnetic wave as a quantum potential barrier $U_g = \hbar\omega_c$. A non-habitual time independent Schrödinger equation, describing guided wave propagation, is established.

Introduction

The association of a quantum potential barrier to a given body with respect to the quantum particle penetrating it or passing in its vicinity has been described in several papers. The notion of quantum potential has first been mentioned by de Broglie [1], Bohm [2] and Philippidis [3]. In these works, the authors use a quantum potential depending on the wave function of the quantum particle.

The concept of a waveguide as a quantum potential barrier is not new. It has been introduced for the first time by Blatt and Weisskopf [4] to explain the mechanism of nuclear reactions. A quantum potential has also been used to describe the interaction of electromagnetic waves with the medium; thus in [5] a Fermi pseudopotential has been used in the Schrödinger equation for solving Maxwell's equations for antennas.

In this work the potential for photons is written as a function of the frequency of the associated waves, and not of their amplitude as in [2] and [3]. We also give an explicit expression for the total energy of the photon in the medium. Thus we have shown that the same object can be described by different quantum potentials for different frequencies of the wave. In this way we obtain a dispersion relation of the potential. Moreover, if we take into account that the guiding effect on the wave disappears for dimension of the guide by an order of magnitude larger than the wave length of the photon, we can state that the quantum potential associated to a body exerts an action up to a distance one order of magnitude larger than the wave length of the incident photon. We are led to the idea that the distance of the interaction photon-object has a dispersive character.

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2. The Waveguide as a Quantum Potential Barrier

The dual character of microparticles and electromagnetic fields is well known. Moreover, in an equal number of cases the physical interaction between the electromagnetic field and the medium is described by wave or corpuscular concept. Usually, for short-wave length electromagnetic fields (γ-rays) we use the corpuscular concept, and for long wavelength electromagnetic fields (radio waves) the wave concept. In microwave circuits all the authors use the wave concept. To test the corpuscular concept in microwave electromagnetic field theory, we start with some very common remarks. The waveguide wall is impenetrable for the electromagnetic field. For a rectangular, waveguide, the photons are placed in an infinite height bidimensional well. A quantum particle in such a potential well cannot be at rest, and also the photons are permanently reflected by the wall of the waveguide. Taking the above remarks for granted, an incident photon normal to the cross section of a rectangular waveguide must change its propagation direction. The energy of the photon must remain unchanged after penetration inside the waveguide (we consider that the waveguide has an infinite mass and that it has per-

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fectly reflecting walls) If we consider now a free space photon of momentum $\boldsymbol{p}_0$ denoting the momentum components inside the waveguide by $\boldsymbol{p}_x, \boldsymbol{p}_y, \boldsymbol{p}_z$, we can write

$$\boldsymbol{p}_0^2 = \boldsymbol{p}_x^2 + \boldsymbol{p}_y^2 + \boldsymbol{p}_z^2 \tag{1}$$

or, dividing by $\hbar^2$, we obtain

$$k_0^2 = k_x^2 + k_y^2 + k_z^2 . \tag{2}$$

The allowed values of wave numbers k_x and k_y are fixed by the transversal dimensions of the rectangular waveguide (a bidimensional (a, b) well of infinite length): $k_x = \frac{m\pi}{a}$ and $k_y = \frac{n\pi}{b}$ (m and n are positive integers). k_z is given by

$$k_z^2 = k_0^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 . \tag{3}$$

In the case of electromagnetic field propagation along the waveguide the axial wavenumber k_z must be real, and consequently $k_0^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$. The critical case $\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = k_c^2$ marks the transition between a progressive and an evanescent wave, respectively, and (3) can be written

$$k_z^2 = k_0^2 - k_c^2 . \tag{4}$$

The equation (4) is obtained in a more complicated way in microwave electromagnetic field theory by solving the Helmholtz equation with boundary conditions for the electromagnetic field.

By introducing the photon energy $E = \hbar\,\omega_0$ and the waveguide potential

$$U_g = \hbar\,\omega_c , \tag{5}$$

the propagation-evanescent condition $k_0 \gtrless k_c$ is seen to correspond to

$$E \gtrless U_g . \tag{6}$$

Thus the constraint on the motion of the photons in the cross section of the waveguide gives rise to a potential barrier as shown in Figure 1. We note this is true under the condition that the wavelength is comparable to the linear dimensions "a" and "b" [6–8].

If we accept, statistically speaking, that a free space incident electromagnetic plane wave is composed of a large number of photons, and accepting the symmetry of the physical system relative to the AA′ and BB′ axis,

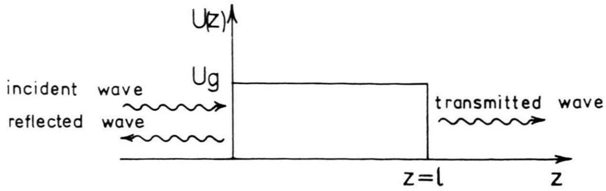

Fig. 1. The waveguide along its axis is analogous to an quantum potential barrier of height U_g.

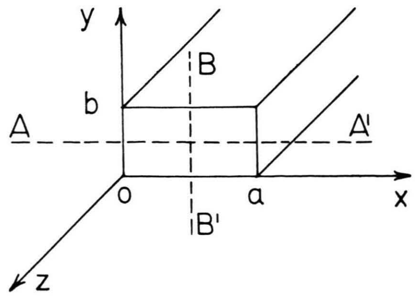

Fig. 2. The cross-section of a rectangular waveguide.

Fig. 2, for normal incidence with respect to the waveguide cross section, an equal number of photons is deviated to the left and to the right, up and down, so a plane electromagnetic wave is decomposed inside the waveguide in four (or two) plane waves.

We shall try further to provide a description analogous to that used in nonrelativistic quantum mechanics of a one-dimensional stationary particle. The quantum mechanical counterpart of (4) is

$$E^2 = p_z^2\,c^2 + U_g^2 , \tag{7}$$

where we used the notation

$$E = \hbar\,k_0\,c,\quad p_z = -i\,h\,\nabla_z,\quad U_g = \hbar\,\omega_c,\quad \omega_c = c\,k_c .$$

The operators in (7) thus lead us to an equation characterizing the propagation of the wave along the z direction:

$$\Delta_z\,\Psi + \frac{1}{\hbar^2\,c^2}(E^2 - U_g^2)\,\Psi = 0 . \tag{8}$$

Hence the term denoted by U_g will be designated as the quantum potential of the medium interacting with the electromagnetic wave [2]. We remark that (8) represents a stationary form of the equation for the waves propagating through a guide having properties of homogeneity in the plane perpendicular to the direction of propagation.

Using (7), the de Broglie relation as well as the relation between the photon energy E and the wavelength in free space we obtain for the refraction index of a medium of quantum potential U_g, the expression

$$n=\sqrt{1-(U_g/E)^2}. \tag{9}$$

3. The Waveguide Filled with Lossless Dielectric Medium

It is interesting to apply (8) to the description of several physical situations. For example, in the case of a lossless plasma of frequency ω_p, by multiplying with $\hbar$ the wave propagation condition $\omega>\omega_p$ one obtains

$$E>U_p. \tag{10}$$

From (9) we obtain the well known relation [9]

$$n_p=\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}. \tag{11}$$

In the same way we obtain the waveguide refraction index

$$n_g=\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}. \tag{12}$$

One may now add two or more potentials (for example a waveguide filled with a plasma or an usual dielectric). It is known that the propagation condition in a waveguide filled with plasma is [9] $\omega_0>\sqrt{\omega_p^2+\omega_c^2}$. By multiplying this relation with $\hbar$ one obtains

$$E>\sqrt{U_p^2+U_g^2}. \tag{13}$$

We can say that the photons interact with a potential barrier of height

$$U_{g+p}^2=U_p^2+U_g^2, \tag{14}$$

or more generally we can write

$$U^2=\sum_i U_i^2. \tag{15}$$

4. Microwave Perturbed Cavity (Nondegeneratetd Case)

Let us examine a parallelepipedical microwave cavity of dimensions, a, b, c, and a dielectric cylinder of radius $R\,(R\ll a, b, d)$ as perturbing element a dielectric cylinder of radius $R(R\ll a,\ b,\ d)$, as perturbing element, as shown in Figure 3. The unperturbed potential is

$$U=\begin{cases}0 & \text{inside the cavity,}\\ \infty & \text{on the walls and outside cavity.}\end{cases} \tag{16}$$

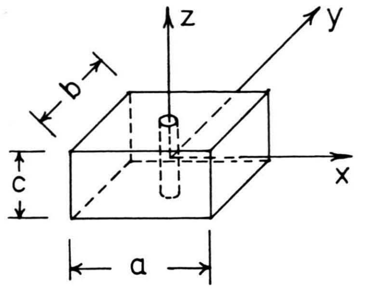

Fig. 3. Rectangular cavity with dielectric cylinder.

The solutions of (8) for the potential U given by (16) are the unperturbed normalized eigenfunctions $\Psi^0_{m,n,p}$:

$$\Psi^0_{mnp}=\sqrt{\frac{8}{abd}}\begin{pmatrix}\sin\\ \cos\end{pmatrix}\frac{m\pi}{a}x\begin{pmatrix}\sin\\ \cos\end{pmatrix}\frac{n\pi}{b}y\begin{pmatrix}\sin\\ \cos\end{pmatrix}\frac{p\pi}{d}z \tag{17}$$

with m, n, p $\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$ and the eigenvalues of energy

$$W^0_{mnp}=\hbar c\pi\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2\right]^{1/2}. \tag{18}$$

If we consider that the perturbating term $W'_{mnp}\ll W^0_{mnp}$, we obtain from (15) the perturbed eigenvalues

$$W_{mnp}\simeq W^0_{mnp}+\frac{1}{2}\frac{W'^2_{mnp}}{W^0_{mnp}}. \tag{19}$$

In a first order approximation, the term W'_{mnp} results [10] from

$$W'^2_{mnp}=\int_{v_{\text{cylinder}}}\Psi^{0*}_{mnp}\,U'^2\,\Psi^0_{mnp}\,dV. \tag{20}$$

According to (9), the cylinder potential

$$U'=E\cdot\sqrt{1-\varepsilon/\varepsilon_0} \tag{21}$$

is constant inside the cylinder, and if we note that the integral is independent of p, we obtain

$$W'^2_{mnp}=E^2(1-\varepsilon/\varepsilon_0)\,Q_{mn}, \tag{22}$$

where Q_{mn} is a constant depending on the cavity dimensions, dielectric cylinder dimensions and cavity mode.

For a cavity resonant, $E\simeq W^0_{mnp}$ and (19) becomes

$$W_{mnp}\simeq W^0_{mnp}\left[1+\frac{1}{2}\left(1-\frac{\varepsilon}{\varepsilon_0}\right)Q_{mn}\right], \tag{23}$$

where the constants Q_{mn} are summarized in Table 1.

Here χ is the relative dielectric volume/cavity volume, i.e. $=\pi R^2/a\,b$.

Table 1.

m	n	Q_{mn}
odd	odd	$4x - m^2 \pi \frac{b}{a} \chi^2 - n^2 \pi \frac{a}{b} \chi^2$
odd	even	$m^2 \pi \frac{b}{a} - \frac{m^2 n^2 \varrho^2}{b} \chi^3$
even	odd	$n^2 \pi \frac{a}{b} \chi^2 - \frac{m^2 n^2 \pi^2}{b} \chi^3$
even	even	$\frac{m^2 n^2 \pi^2}{b} \chi^3$

Using only the first term in the expression of Q_{11}, the relative frequency variation of the fundamental cavity mode is

$$\frac{\Delta\omega}{\omega_0} = \frac{2\pi R^2}{a\,b}\left(1 - \frac{\varepsilon}{\varepsilon_0}\right). \tag{24}$$

The Eq. (24) is in agreement with the usual electromagnetic theory [11]. It can be seen from Table 1 that there is a dependence of the perturbation on the parity of the integers m and n.

5. Conclusions

The electrodynamc characterization of wave processes in transmission lines in terms of the vector functions $\boldsymbol{E}(u_1, u_2) \cdot \exp(\mp\gamma z)$ and $\boldsymbol{H}(u_1, u_2) \cdot \exp(\mp\gamma z)$ (where u_1, u_2 are the coordinates in the cross-section) carries much more information than may be necessary for the design of a microwave system and also in microwave measurements. Indeed, we are interested in the power to be transmitted, the relative magnitude of the incident and reflected waves and the phase shift and attenuation over a definite length of the transmission line. The power, phase shift and reflection coefficient can be found by experiment, whereas the measurement of the field components and their distribution function entails appreciable difficulties.

The major result reported in this paper is that a propagation medium, a microwave circuit, a dielectric or a plasma medium can be considered as a quantum potential barrier for the electromagnetic wave. The main advantage of a quantum formalism consists in the ability of performing the guided wave analysis onto a single particle, i.e. a photon instead of using Maxwell's field equation. By this method we performed the analysis of directional couplers, a periodic structure in the waveguide, a microwave interaction with anistropic medium a.s.o. [12].

We note that (1) is generated by the wave equation by supposing a plane structure for its solution. The vectorial properties of the e.m. field against Lorentz transformation is preserved by our formalism. Indeed, by means of (8) we determine the energy E characterizing the propagation along the z direction. The energy of this mode is influenced by the transversal modes through the potential term U_g involved in (8). Therefore the reflection on the guide walls yields a coupling between the transversal modes on one hand and the longitudinal one, on the other hand. Further, (7) defines the momentum p_z. In this way the 4 components p_x, p_y, p_z, E of the fourdimensional p^μ are given in a quantized form, and consequently the plane solution of the wave equation is discretized.

Our procedure describes microwave propagation by replacing the problem of solving the wave equation with specific boundary conditions by an eigenvalue problem of the Schrödinger type. The advantage of the latter procedure over the former one consists in its simplicity. The transfer of a formalism from one branch of physics to another one is a frequent practice. Indeed the specific methods of e.m. field theory ae often used for solving elegantly some quantum mechanical problems. Among many examples are the path integral method [13], the formalism developed by Albeverio et al. [14] in connection with some solvable models in quantum mechanics, the treatment of electron propagation in microelectronic devices [15].

Various aspects of microwave propagation have previously been investigated by standard methods of e.m. field theory. Thus the reflection by an obstacle was treated by Boström in [16, 17], while the theory of resonant cavities is given in [18].

The extension of this formalism to the description of the microwave propagation in a guide having a more complex profile, is in progress.

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